Visual Mathematics Current Research, Sensual as Hollywood Cinema

Splendid Ideas, Brilliantly Converted into Pictures

Mathematics can be pictorial. Computers can contribute. Are there pictures which are proofs?

BY GERO VON RANDOW
TRANSLATED BY ROLAND GIRGENSOHN

Pictures have suggestive power. This is also true in the natural sciences with their particle tracks, molecular models and genealogical trees. How about Mathematics? What gets drawn on blackboards are not always just mathematical symbols (which however may be iconic, that is, small figures which resemble what they signify, such as the sign > for the "greater than" relation). Trees can be seen as well, or nets, or strange shapes with handles. Other mathematicians crouch in front of the monitor and play around with colourful pictures.

In doing so, they apply instruments to their science which arose out of just this science. Mathematically supported visualization has grown to be a powerful tool in technology and science. Designers of automobiles, climate scientists, or producers of microchips, all transform their models into moving simulations and thus gain insight. They profit from investments made by the film and game industry, which has hired some of the best mathematicians to develop realistic looking picture sequences on their computers. The breakthrough was the 1993 movie "Jurassic Park" where computer-generated sequences, which looked almost real, were meshed with conventionally filmed scenes. There was also a mathematician in the movie. A dinosaur bit him.

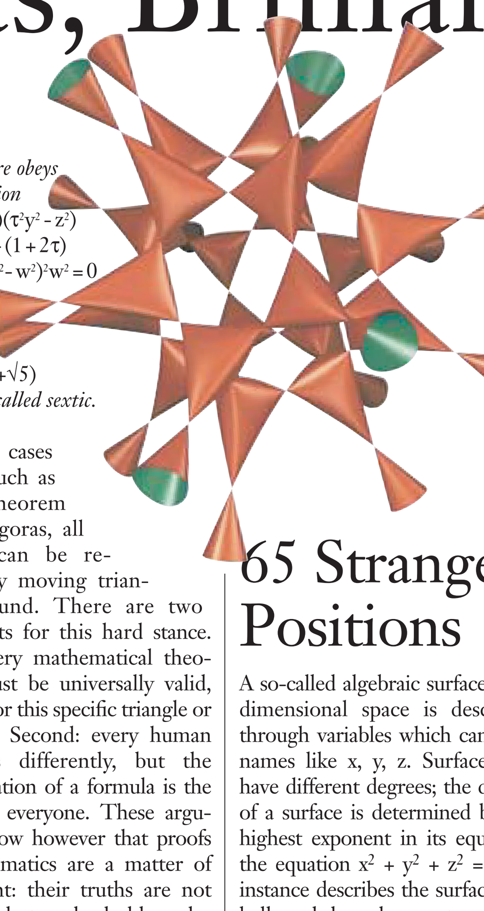
The first crisis in mathematics came from a visual problem. The Pythagoreans believed that all laws in the cosmos are governed by the integers. But then they discovered that the basis of a right-angled triangle with sides of length 1 must have a length of $\sqrt{2}$: a number which cannot be expressed as a ratio of two integers. It was visually clear to them, however, that such a triangle existed.

Arguments based on pictures can often be found in the ancient texts, for example in Euclid (ca. 300BC), the founder of strictly formal geometry. He had trouble to define what, exactly, a point is: something that has no dimensions. But then, how can something like that lie on a line, and what does "lie" mean in this context? The solution was found in the 17th century: René Descartes defined a point through two numbers (its coordinates on the x- and y-axis). This was of course not the end of pictures in mathematics: suddenly many mathematical relationships could be visualized in a coordinate system.

In mathematics there are always opposing trends: rigorous arguments vanquish visual ones, and not long thereafter new objects arise to be visualized. In the 19th century geometries were developed where parallels intersect, or spaces with four, five, arbitrarily many or infinitely many dimensions. They can be accurately described only in a formal manner. But shortly after that topology began to flourish. It started with the concrete problem of classifying shapes by those properties which remain unchanged under continuous distortion. Since then topology has lost much of this visual quality – but nowadays there is so-called graph theory where diverse trees and nets are drawn. Even in the most abstract branches of mathematics, e.g. in algebra, the visual is present. Its "groups", "rings" or, in general, notions of symmetry reflect that mathematicians often think of forms when they write formulas.

Pictures generated by computers can aid in getting ideas. Bizarre curves can often point to something that is mathematically interesting. Honest mathematical experiments can be carried out using visualization by computer programs. And pictures are suited for communication, most importantly for teaching, provided they do not replace the understanding of the underlying formulas.

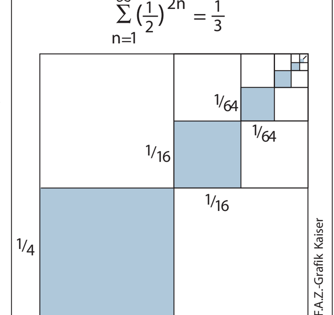
Can pictures be more than visualization? Do they just make a thought plausible, or can they be conclusive? After all, they can refute a conjecture, for example on the shape of a curve. They can perfectly well act as proofs: namely in those cases where there is agreement that a visually evident fact can also be proved rigorously. Almost all mathematicians, however, reject the idea that pictures can be true proofs – even in those cases where, such as in the theorem of Pythagoras, all doubts can be resolved by moving triangles around. There are two arguments for this hard stance. First: every mathematical theorem must be universally valid, not just for this specific triangle or parabola. Second: every human visualizes differently, but the interpretation of a formula is the same for everyone. These arguments show however that proofs in mathematics are a matter of agreement: their truths are not absolute but only hold under certain human-made rules – but then with certainty.



This figure obeys the equation
$4(\tau^2x^2-y^2)(\tau^2y^2-z^2)(\tau^2z^2-x^2)-(1+2\tau)(x^2+y^2+z^2-w^2)^2w^2=0$
where
$\tau=1/2(1+\sqrt{5})$
It is a so-called sextic.

Visual Proof of the Assertion

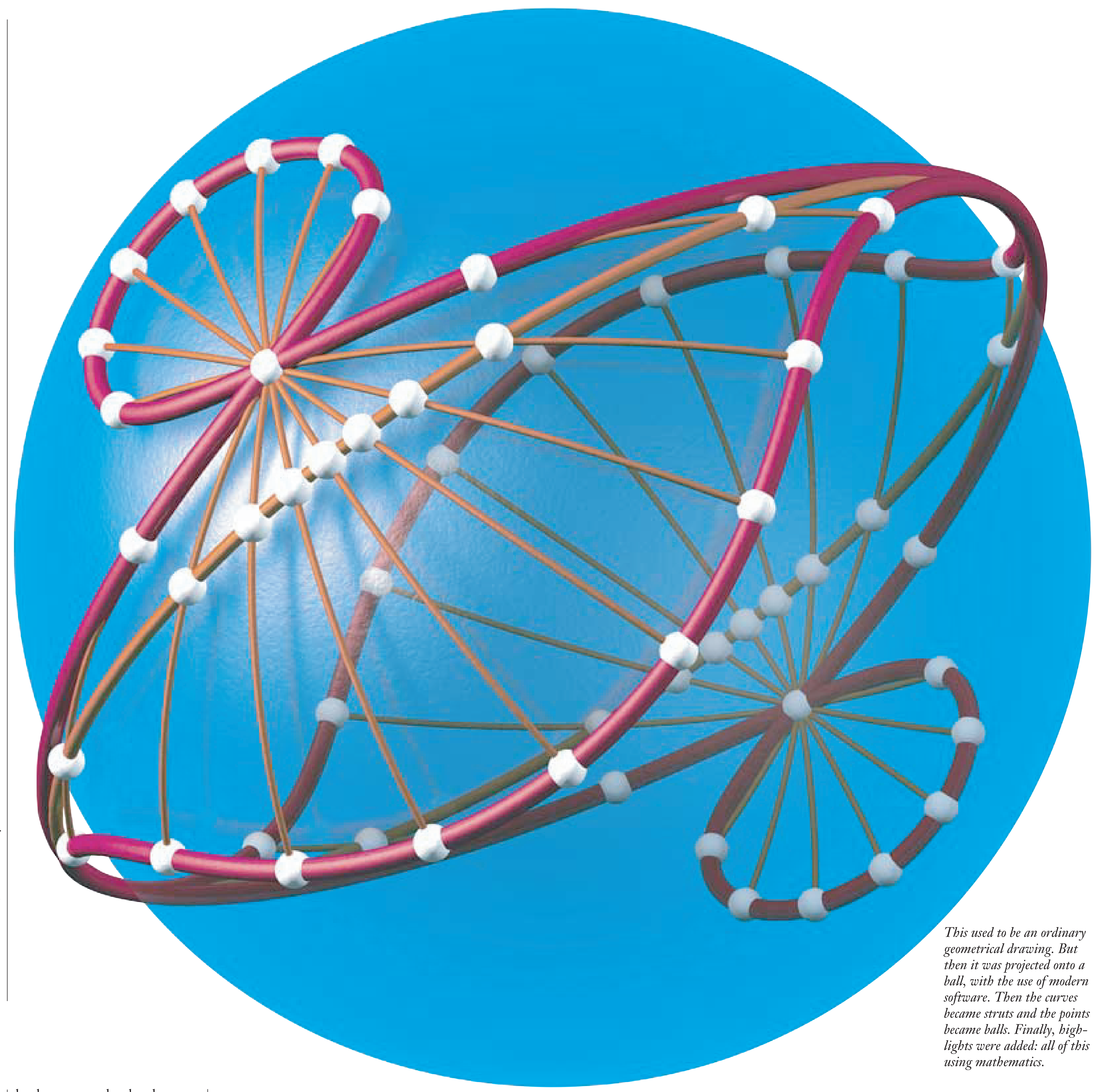
$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{2n}=\frac{1}{3}$$



The Greek summation sign instructs us to let n grow from 1 to infinity, to use these numbers to exponentiate $(1/2)^{2n}$, and to add the results. The sum approaches the value 1/3. The picture suggests that this assertion is true. Is this already a proof? And if so: are visual proofs only suited to really simple cases like this one?

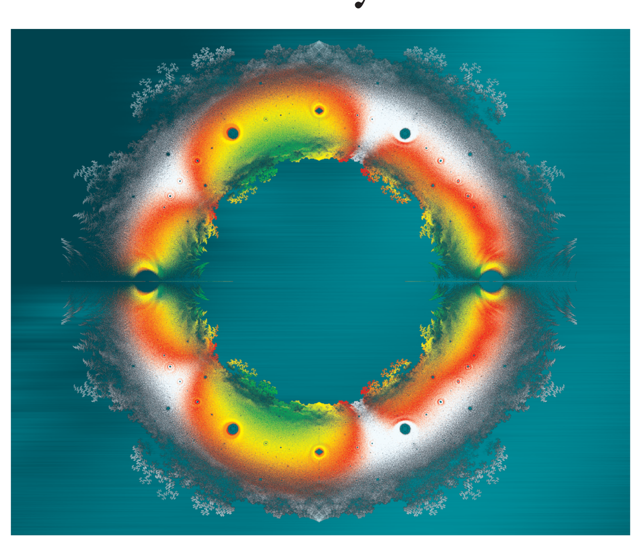
65 Strange Positions

A so-called algebraic surface in 3-dimensional space is described through variables which can have names like x, y, z. Surfaces can have different degrees; the degree of a surface is determined by the highest exponent in its equation: the equation $x^2 + y^2 + z^2 = 1$ for instance describes the surface of a ball and has degree two. The surface in this picture has degree six, since if its equation is multiplied out, then w^6 appears. Therefore it is called a sextic. There are 65 places on the surface where things become interesting. They are so-called "double points": they behave similarly to those points on a curve which do not admit a well-defined tangent, as for example pointed corners. Double points of a surface do not admit a tangent plane. This picture pertains to a research topic which already emerged at the beginning of the 20th century, but only now gains steam: what is the maximal number m of double points of an algebraic surface of degree d in three-dimensional space? For degrees three to six there are established theorems, for higher degrees there are only certain intervals known in which to look for m. The 65 double points of this sextic is the maximal number m. It has the shape of an icosahedron; one would like to know why. *by GvR*



This used to be an ordinary geometrical drawing. But then it was projected onto a ball, with the use of modern software. Then the curves became struts and the points became balls. Finally, highlights were added: all of this using mathematics.

Zeros of a Polynomial



Red indicates where the depicted function behaves especially stably when a certain element in it is changed.

This is not a wreath but an example for how even relatively simple mathematical objects can blossom when one uses a computer to release them into the plane of complex numbers. Complex numbers are in some sense a two-dimensional extension of the usual (real) numbers. They are needed to provide roots of negative numbers. All complex numbers z lie on a plane, and one can apply functions to them, for example polynomials, in this case it is $f(z)=a_0+a_1z+a_2z^2+a_3z^3+\ldots+a_{21}z^{21}$.

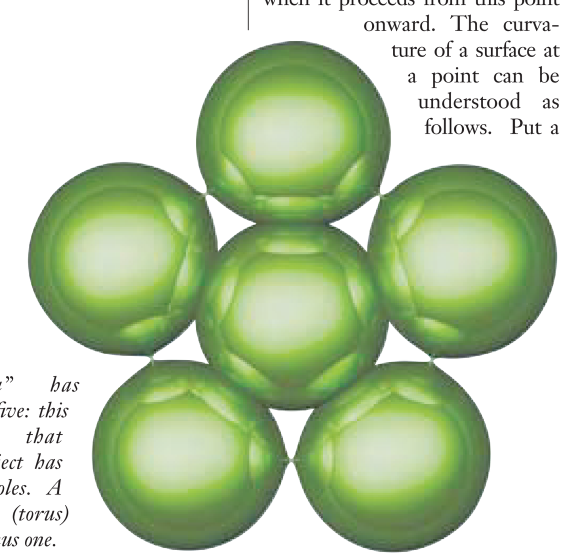
The picture shows the zeros of this function, that is, all numbers z for which $f(z) = 0$, where the coefficients a_0 to a_{21} take both the values -1 and +1 – with the exception of a_3. The colour indicates how sensitive the position of the zeros depends on deviations of the coefficient a_3 from the value 1. *by Ulf von Rauchhaupt*

Does this Figure Exist?

This formation of green balls, called "Penta", satisfies certain requirements for its geometrical properties. Interestingly, symmetry is not one of them. It has been found through complicated trials on the computer. Since it does not come from a systematic construction, the picture only suggests its mathematical existence, nothing more: it is no proof of existence. Therefore it is possible that Penta does not exist in a mathematical sense; it is only known that objects of this type exist generally.

These are the criteria: firstly it is a closed surface in three-dimensional space, therefore has no boundary. Secondly it only has a few holes, but more than one. Thirdly it has "constant mean curvature" – to explain what this is it is necessary to be a bit more detailed. The curvature of a curve at a point results from the rate of change the tangent undergoes when it proceeds from this point onward. The curvature of a surface at a point can be understood as follows. Put a book on your head; when you move it around, you will see if your head is rather peaked or rather round. Mathematics specifies more precisely the curvature of a surface at a point P: first construct a perpendicular line to the surface at P. Then consider all planes that contain this line. The planes will intersect the surface in a curve. Each of these curves will have a certain curvature at P. Now consider the two curves with minimal and maximal curvature. Half the sum of these values in P is the "mean curvature" of the surface in this point. Its name is H.

This H is an important quantity for so-called minimal surfaces: they are surfaces which have minimal size under constant constraints (e.g., a certain boundary or a certain enclosed volume). There is a proof by Euler that says that H is zero for all minimal surfaces with boundary; the neighbourhood of every point then is either completely flat or shaped like a saddle, in this case minimal and maximal curvature add to zero. The surfaces of spheres, for example soap bubbles, are closed (i.e., without boundary) minimal surfaces which enclose a given volume; at every point they have a constant and positive H. Penta has a constant mean curvature H as well and is a minimal surface for a given volume if the number of holes, 5, is prescribed. *by GvR*



"Penta" has genus five: this means that the object has five holes. A donut (torus) has genus one.

A Point Goes Hiking

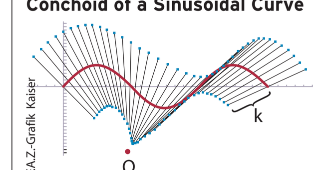
The structure above is easy to construct – the rationale is the underlying software. What shines there so nicely is the "conchoid of a certain curve, projected onto the surface of a sphere", and it is constructed as follows: at first a curve is drawn, let us call it C. Then a point O outside of C is chosen. We may call it the "pole". Furthermore we choose a number. Call it k: it is a "constant". Now we draw a straight line through O which intersects C somewhere; the intersection is named P. Then points Q1 and Q2 are determined: those points on the line which have a distance of k to P. Now everything is prepared: the conchoid is generated by rotating the line around the pole, so that the point P goes hiking. The points Q1 and Q2 draw the conchoid (see the figure to the right); mathematicians write: "The conchoid from O to the curve C is the position of all Q1 and Q2 for variable P and constant k."

Using the geometry software, "Cinderella" even math-laymen can construct conchoids and other objects. The tour of the point P can even be shown on the move, this is then called "dynamic geometry". Using Cinderella it is easy to project the conchoid, lying in the plane, onto the surface of a sphere – as in the picture above. Cinderella also provides other projections. That is, it is possible to switch from Euclidean to different non-Euclidean geometries, thus to work in the plane as well as on spheres and other shapes.

The reason why this is so easy is that the program performs its internal computations on the basis of a theory from the late 19th century which unifies all of these geometries; shown, however, is only that representation chosen by the user. The clever program in the background also avoids some unpleasantnesses known from other geometry programs – such as that some point in a construction cannot be moved continuously but only in steps where mathematically ambiguous situations occur; in such cases the program makes a mathematically justified decision.

Cinderella can be viewed at www.cinderella.de and is also suited for schools. *by GvR*

Conchoid of a Sinusoidal Curve

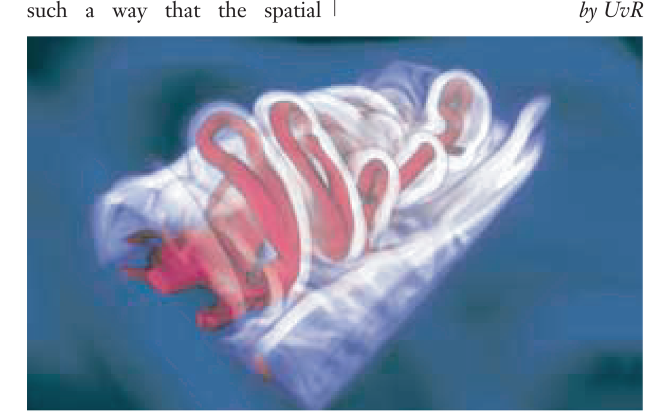


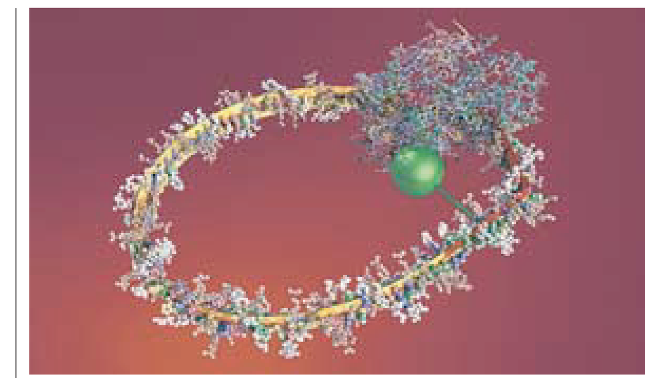
The point O is the point from which the conchoid is constructed.

Shocks Make Waves

Hydrodynamics is the physical theory of flows. No matter if you want to optimize car bodies or understand cosmic nebulas, you will deal with hydrodynamic equations – and only on rare occasions can they be solved without a computer. Nowadays there exists refined software to compute complicated phenomena such as three-dimensional shock fronts, turbulence, or – as in the picture below – combinations of both. The shape shows what happens when a cylindrical shock-front vertically hits a likewise cylindrical region of lower density and thereby generates tubes of rotating gas (in red), that is, turbulences. It is not easy to compute something like that. It is also difficult to represent the results in such a way that the spatial structures so generated are recognizable.

One possibility is to colour and shade those surfaces in space on which the quantities we are interested in (such as the pressure of the gas) take given values. One disadvantage of this is that it takes a lot of effort to compute three-dimensional surfaces. Also, such pictures can easily become too complex. For the picture below so-called "volume rendering" was used, which means that the scene is subdivided into small elements ("voxels"). The physical quantities in each voxel determine its colour and, more importantly, its transparency, so that entangled structures like the one shown here can be discerned. The process of visualization is now even interactive. *by UvR*



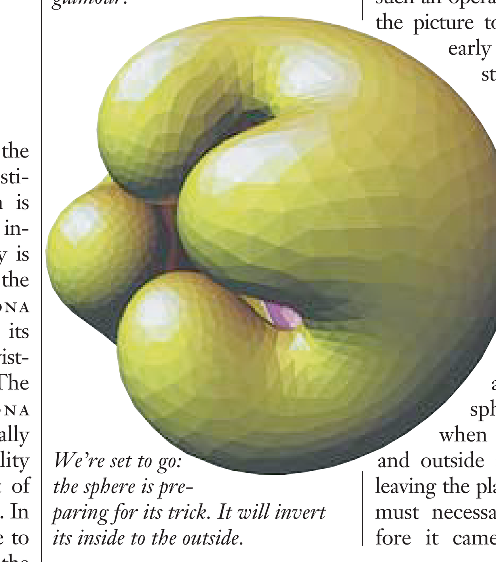


Glittering DNA Molecule

The picture editors of scientific journals love this: DNA with gold and glamour.

Big biomolecules are among the most complex systems investigated by science. Among them is the DNA molecule, where all information about a living body is stored. It is conjectured that the biochemical properties of a DNA molecule are determined by its behaviour under flexing and twisting, among other things. The segments of a single DNA molecule are not all equally flexible, but their flexibility depends on the arrangement of the base pairs at the given spot. In principle, it should be possible to compute the flexibility from the chemical properties of the individual atoms, but this is asking too much of even the fastest supercomputers available today. A better approach therefore is to model such a DNA molecule as a piece of wire whose elasticity varies continuously from place to place. Once such a wire with varying flexibility is specified, its properties can be studied on the computer and it can be investigated as to how the mechanical behaviour of the DNA changes when a protein molecule (green ball) is attached to it. Pictures, generated by the computer, like this one help to visualize such configurations – and look good on the title pages of scientific journals. *by UvR*

How to Turn a Sphere Inside Out



We're set to go: the sphere is preparing for its trick. It will invert its inside to the outside.

A sock with one open end can be turned inside out, a tightly sewn soccer ball cannot. This is in contrast to the mathematical model of a soccer ball, the sphere. It can indeed be deformed continuously without tears and kinks in such a way that in the end its inner surface will be the outer surface – but only if the surface of the sphere is allowed to penetrate itself during the procedure. Topologists call such an operation "eversion", and the picture to the left shows an early stage: in the next stage the four finger-like appendages will penetrate each other and will in this way draw the inner surface of the sphere to the outside. This would be impossible with the two-dimensional analogue of the sphere, the circle: when exchanging inside and outside of a circle without leaving the plane, cracks and tears must necessarily occur. Therefore it came as a surprise to mathematicians when in 1957 a doctoral student named Stephen Smale proved that eversion of spheres is possible. Since then mathematicians have tried to comprehend such operations via sketches on paper or by using wire models. The simplest eversions possible, however, were found in the nineties by John Sullivan and George Francis from the University of Illinois using the computer. They modeled the surface of the sphere as consisting of rubber which tautens on deformation and therefore becomes charged with energy. They knew from theoretical considerations the shape of a sphere that is half-way everted and has maximal energy. Then they used the computer to calculate how this bizarre entity contracts to a figure with minimal energy – which is just a sphere – and so they got pictures of each stage of the eversion. It was not at all clear a priori whether a half-everted sphere with maximal energy really shrinks down to an inside-out sphere by itself. Only the experiment on the computer showed that it worked. Was this already a proof? *by UvR*

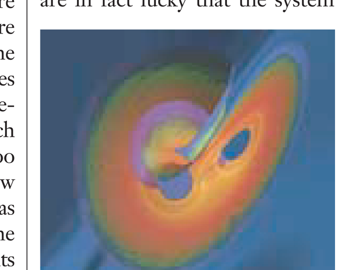
Pictures from Sound

Sound waves are not only useful to transmit speech and music. Many different branches of science – from ultrasonic medical diagnostics to research of earthquakes – use the ability of sound to carry information about its source. Explorations using sound flourished with the advent of fast computers and efficient algorithms. This is because diagnostics with sound are mathematically finicky. It is a relatively easy exercise to send a simple sound wave against an object of known form and then to compute the shape of the reflected waves. If, however, the object has a complicated three-dimensional structure – such as the dolphin here in the picture – then the underlying equations can only be solved approximately. In practice, however, the inverse problem is of more interest: one receives sound waves and wants to draw conclusions about the shape of the object on which they were reflected. Solutions to such inverse problems have the unpleasant property that they are very sensitive to disturbances and usually have several solutions: one and the same pattern of sound waves can be generated by different objects. Mathematicians call this an "ill posed problem". The reflecting object becomes unique only if one manages to extract additional information from the equations. Often this is done by viewing the object from different angles and superposing the received sound waves in the computer. *by UvR*

Submarine or dolphin? The reflected sound waves betray it.

Chaos with Structure

This picture is something of an icon for a sector of research which was "in" in the late eighties – not coincidentally just at the time when computers became affordable. This "in" sector is the dynamics of nonlinear systems, aka chaos theory. The behaviour of such a system often delicately depends on its initial state – in contrast to linear systems, such as two celestial bodies which orbit around each other because of gravitational attraction. Systems like this always repeat their constellations and can therefore be predicted with precision. We are in fact lucky that the system consisting of the Earth and the Sun is very close to an ideal linear system. In contrast to that, the orbits of nonlinear systems are usually not closed and are often similar to the diagram shown here. The underlying system was discovered by the meteorologist Edward Lorenz, when he formulated a simple model for the flow of gas. The picture shows that although the curve never returns to itself, it also does not move completely erratically through space. Instead it forms a strange pattern – an "attractor" (see www.wam.umd.edu/~petersd/lorenz.html). Often attractors have interesting properties which can provide information about significant aspects of the long-term development of the underlying dynamical system. Modern techniques allow investigation of such properties of the attractor without having to calculate the complete temporal development of the system. Often visualization of such properties, e.g. by colour coding, is helpful. *by UvR*

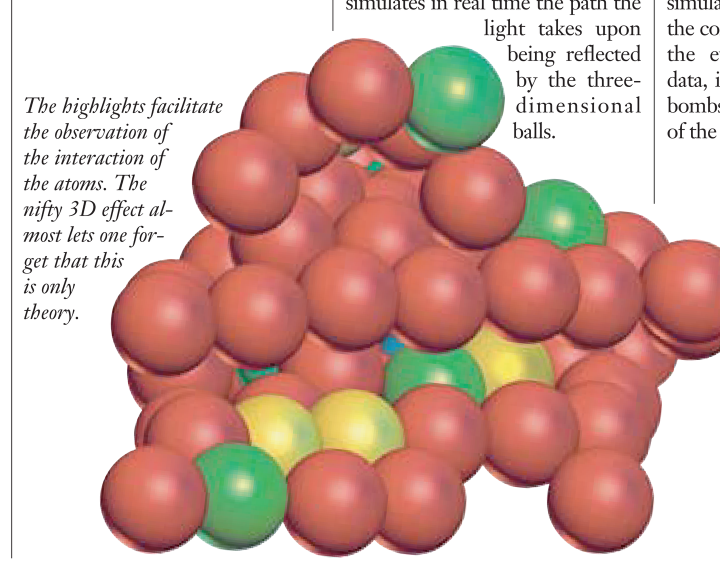


The "Lorenz Attractor", once a popular motif on t-shirts and coffee cups.

35 Million Balls

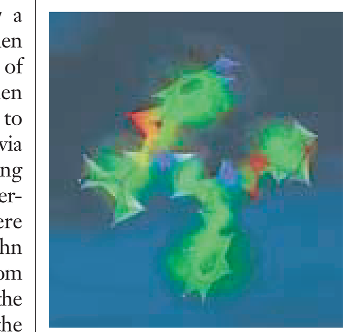
This is a minuscule clipping from a picture with 35 million balls; the supercomputer in Los Alamos which was used to generate it presents ten to fifteen such pictures per second in an interactive visualization. The pictures show the spreading of cracks; the balls represent atoms. The photorealistic effect is achieved by a technique called "ray tracing": the program simulates in real time the path the light takes upon being reflected by the three-dimensional balls.

Such visualizations are only possible using computers which are massively parallel, that is, where many processors work at the same time. To govern this teamwork is a mathematical problem in itself: find, for a given task, with given resources of hardware, and for a given timeframe, an optimal segmentation into subtasks. Such massively parallel computations are performed in simulations of the climate and in the construction of aircraft, also in the evaluation of astronomical data, in the development of atom bombs, or in the computer studios of the film industry. *by GvR*



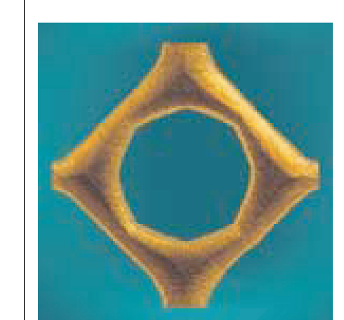
The highlights facilitate the observation of the interaction of the atoms. The nifty 3D effect almost lets one forget that this is only theory.

Molecular Ballet



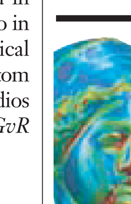
Often the function of a biomolecule is determined by the different stable forms it can achieve. As soon as its principal features are known, its favorite forms ("conformations") can be computed. The mathematical formulation of the problem can be traced back to concepts, some of which are around 170 years old, and to the work of a multitude of physicists in the 20th century. But it is not enough to formulate the problem – it should also be possible to compute its solutions. In the case shown here tricky transformations of the problem and the latest computing techniques were necessary. The succession of the different conformations of the molecule was estimated with the use of new techniques which go back to a method (Markov Chains) which is more than 100 years old; the estimates were prepared using an artificial neural net, which is a technique from the area of artificial intelligence. Every contiguous pair of links is represented as a triangle. The glowing nebulas represent the probabilities that the molecule assumes a certain shape. *by GvR*

Virtual Etching



Stone tools were picked and battered into shape; metal was smelted, founded and forged. Silicon however, the material on which the information age is based, is crystallized and etched. The smaller the transistors on the semiconductor chips become, the more important accurate etching is.

The Finnish scientists who produced this picture have simulated this important process on a computer within the precision of one atom. They investigated how the probability with which a corrosive chemical peels away a silicon atom from the crystal depends on the atomic neighbourhood. They have prepared their results graphically in such a way that it is possible to compare them directly to microscopic photographs of real etching patterns. *by UvR*



The Duisburg mathematician Martin Rumpf is one of the most sought-after computer graphics experts of the world. His methods can be used to quickly depict even the most complicated surfaces of creatures, such as for example this head. Three years ago he and his colleagues Konrad Polthier and Hans-Christian Hege set the ball rolling: during a conference on mathematical visualization they made the suggestion to combine pictures from current research areas into a calendar. Now it is finished (H.-C. Hege, K. Polthier, M. Rumpf: **MathInsight 2002**, Springer-Verlag, DM 49, 49). The pictures on these pages are taken from it. Not long before that, a compendium of current research was published which addresses the layman with some mathematical training: **Mathematics Unlimited – 2001 and beyond**, written by the cream of the field, it details on 1237 pages those areas where important work is done nowadays (more information under www.springer.de).